GAUSS Seminar

3/16/2012

Topics

- 5:30-6:00: Calculus - Lagrange multiplier method in solving constrained optimization problems
- 6:00-6:20: Problems session
- 6:20-7:00: Linear Algebra – Linear Representation, Eigenvalues and eigenvectors, diagonalization, SVD, PCA

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Calculus

- A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

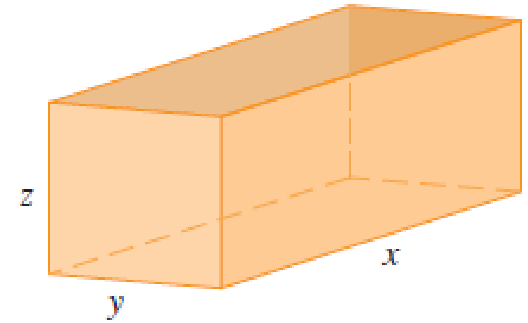

FIGURE 10

SOLUTION Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 10. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy \frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V, or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V, so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m^3.

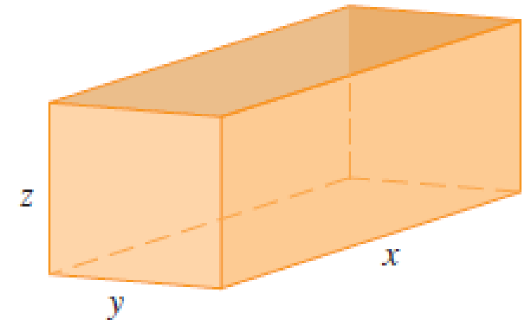

FIGURE 10

SOLUTION Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 10. Then the volume of the box is

$$V = xyz \longrightarrow \text{minimum}$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

subject to the constraint: $2xz + 2yz + xy = 12$

Solving this equation for z, we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy\,\frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

Let's see how we can do this without solving explicitly for z :

Let $f(x, y, z) = xyz$ and $g(x, y, z) = 2xz + 2yz + xy - 12$. Then the problem we are trying to solve is:

$$\text{minimize V} = f(x, y, z) = xyz$$
$$\text{subject to } g(x, y, z) = 2xz + 2yz + xy - 12=0.$$

Pretend that we solved z from $g(x, y, z)$=0, obtaining $z = h(x, y)$. Then

$$V = f(x, y, h(x, y))$$

So,

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x} f\big(x, y, h(x, y)\big) = f_x\big(x, y, h(x, y)\big) + f_z\big(x, y, h(x, y)\big)h_x(x, y)$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y} f\big(x, y, h(x, y)\big) = f_y\big(x, y, h(x, y)\big) + f_z\big(x, y, h(x, y)\big)h_y(x, y)$$

We need to calculate h_x and h_y in

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x} f\big(x, y, h(x, y)\big) = f_x\big(x, y, h(x, y)\big) + f_z\big(x, y, h(x, y)\big)h_x(x, y)$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y} f\big(x, y, h(x, y)\big) = f_y\big(x, y, h(x, y)\big) + f_z\big(x, y, h(x, y)\big)h_y(x, y)$$

Note that $z = h(x, y)$ satisfies the equation

$g\big(x, y, h(x, y)\big) = 0$ for a collection of (x, y)

Use implicit differentiation to get

$$\frac{\partial}{\partial x}\, g\big(x, y, h(x, y)\big) = 0$$

Or

$$g_x\big(x, y, h(x, y)\big) + g_z\big(x, y, h(x, y)\big)h_x(x, y) = 0.$$

So,

$$h_x(x, y) = -\frac{g_x\big(x, y, h(x, y)\big)}{g_z(x, y, h(x, y))} = -\frac{g_x}{g_z}.$$

Similarly,

$$h_y(x, y) = -\frac{g_y\big(x, y, h(x, y)\big)}{g_z(x, y, h(x, y))} = -\frac{g_y}{g_z}.$$

Substitute g_x and g_y by these formula in $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$, we get

$$\frac{\partial V}{\partial x} = f_x + f_z\left(-\frac{g_x}{g_z}\right)$$
$$\frac{\partial V}{\partial y} = f_y + f_z\left(-\frac{g_y}{g_z}\right)$$

Finally, the stationary points are solved from the system

$$f_x + f_z\left(-\frac{g_x}{g_z}\right) = 0, f_y + f_z\left(-\frac{g_y}{g_z}\right) = 0.$$

Solving these for x and y to find the minimum point as in the example.

What's the general pattern?

Note that

$$\frac{f_x}{g_x} = \frac{f_z}{g_z} = \frac{f_z}{g_z} = \frac{f_y}{g_y} = \lambda$$

So, indeed, we are solving equations

$$f_x - \lambda g_x = 0$$
$$f_y - \lambda g_y = 0$$
$$f_z - \lambda g_z = 0$$

Lagrange Multiplier Method

- Let $L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$
- To find the possible minimum (or maximum) points, set the partials of L to 0 and solve the system for x, y, z, λ.
- Lagrange Theorem says these (x, y, z) are also candidates for the extreme points for the constrained optimization problem:

$$f(x, y, z) \rightarrow max$$
$$subj. to\ g(x, y, z) = 0$$

Practice Problems:

1. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

2. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where p, q, and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most $\frac{2}{3}$.

3. Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leqslant 1$.

4. Find the maximum and minimum values of the given function under the given constraint.

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n;$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

5. Find the maximum and minimum values of $f(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 + x_3^2$ *subject to* $x_1^2 + x_2^2 + x_3^2 = 1$.

How could Matlab help?

- ID and 2D plots: plot, plot3
- surf, mesh, scatter, contour, contour3, …
- Optimization toolbox
- Image processing toolbox

Linear Algebra Review

- Vectors: $\vec{\boldsymbol{v}}, \boldsymbol{v}, v, u, \ldots,$
- Matrices: $A, B, M, X, W, U, V, \ldots$
- Scalars: $a, b, c_1, k_1, x_1, \ldots$
- Euclidean space: R^n
- Space of matrices of size $m \times n$: $R^{m \times n}$
- The set of vectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_{\boldsymbol{2}}, \ldots, \vec{\boldsymbol{v}}_{\boldsymbol{k}}$ is linearly independent if

$$a_1\overrightarrow{\boldsymbol{v}_1} + a_2\overrightarrow{\boldsymbol{v}_2} + \cdots + a_k\overrightarrow{\boldsymbol{v}_k} = 0 \underset{implies}{\Longrightarrow} a_1 = a_2 = \cdots = a_k = 0.$$

Linear Algebra Review, 2

- Basis of $R^n =$ set of linearly independent vectors whose span is R^n
- Linear independence = no redundence
- span = collection of all linear combinations
- Standard basis: $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ for R^n, where

$$\vec{e}_j = [0, \dots, 1, 0, \dots, 0],$$
$$\uparrow j - th\ place$$

- Other basis: the columns of a $n \times n$ non-singular matrix $M = [v_1, \dots, v_n]$

Column View of Matrices

- A matrix is a (ordered) collection of column vectors (its columns).
- Let $W = [w_1, w_2, \ldots, w_k] \in R^{n \times k}$, as a "vector" of its columns $w_1, w_2, \ldots, w_k \in R^n$.
- Then the following are the same: For $a \in R^k$

$$\text{(matrix multiplied by vector) } Wa$$

and

(linear combination) $a_1 w_1 + a_2 w_2 + \ldots + a_k w_k$

Represent Matrix Using Its Column Basis

- Assume $D \in R^{m \times n}$ and m and n are a "large" number (like 10^6).
- How many entries in D?
- Answer: $m \times n$
- Takes a lot of memory to store D (in computer)
- What if we know that the columns of D come from a much smaller k dimensional subspace of R^m? Say $k = 6$.
- We can use this to "identify" each column of D.

Dimension Reduction

- Let $U = [u_1, u_2, \dots, u_6]$ be the matrix formed by a basis of the column vectors of D.
- So, there is a matrix V such that $D = [d_1, d_2, \dots, d_n] = [u_1, u_2, \dots, u_6]V$ (Why?)
- What the dimension of V?
- Answer: $6 \times n$
- How to find U and V?
- (to be continued)

Assignments

- Finish all five problems at the end of the discussion of Lagrange multiplier method.
- Use Matlab everyday (at least check out all the functions I mentioned on the slide titled "How can Matlab Help?"
- Read the slide titled " Dimension Reduction" carefully.